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Escape Statistics for Systems Driven by Dichotomous Noise. I. General Theory

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Previous results on first-passage-time statistics for systems driven by dichotomous noise are extended in order to cover the escape from regions including fixed points of the stochastic flow. For such regions a treatment splitting the escape through one or the other boundary is required. The obtained escape probabilities and mean exit times are relevant for the complete characterization of stochastic systems undergoing bifurcations.

KEY WORDS: Escape probabilities; mean first passage times; dichotomous noise.

1. INTRODUCTION

The most interesting aspect in the analysis of nonlinear dynamical systems is the study of the asymptotic, i.e., long-time-limit, behavior. If one considers that, in a first approach, the system is deterministic, bifurcation theory[(1)] can be used to elucidate the number, characteristics, and stability of the asymptotic solutions. Once the bifurcation diagram has been obtained one can discuss the evolution of the system in terms of the initial conditions. In the more realistic stochastic situation, when one or more of the parameters fluctuate, the long-time behavior is usually obtained by calculating, when this is possible, the stationary probability density for the relevant quantities (see, however, Hasmiskii[(2)] for an approach based on

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Lyapunov functions, or Lücke[3] for another based on the moments of the process itself). At this point different authors use different concepts to describe the stochastic analogs of the deterministic bifurcations, e.g., analysis of the moments of the distribution[4] or changes in the shape or the extrema of the stationary density,[5–8] or even by defining a stochastic bifurcation region.[9, 10]

However, the evolution of a stochastic system is richer than that of the deterministic counterpart, and a complete understanding of the stochastic behavior cannot be obtained by simply looking at stationary properties. The fact that a stochastic system can explore different regions may have important consequences in the temporal evolution of the process. Therefore, a knowledge of both large-deviations properties and local characteristics, i.e., exit statistics, escape probabilities, and mean first passage times (MFPT), and stationary probability densities, is necessary in order to obtain a full description of stochastic systems. This is particularly clear when the noisy parameter can only take values in a finite interval, as many occur in an experimental situation, and the system can enter a region without possibility of return.[1] If this is the case, the analysis of the probability of escape from a given region and the mean time to escape is essential to characterize fully the time evolution of the process.

With the above considerations in mind, in this work we study the escape probabilities and MFPT of arbitrary one-dimensional nonlinear systems perturbed by dichotomous noise (only two values allowed for the noise). The interest of such systems arises in situations where the correlation time of the stochastic perturbation is not negligible and/or the noise is known to be bounded. As for the latter condition, actual fluctuations may be bounded for physical reasons, and it has been found[12] that perturbations taking values in a finite range and acting on a bifurcation parameter bring about qualitatively different behavior from that of a system driven by an unbounded, e.g., Gaussian, noise (see also ref. 13 for other implications of bounded noises).

Whereas the MFPT can be exactly calculated for one-dimensional systems driven by Gaussian white noise, the problem for non-Markovian processes, i.e., colored noise perturbations, remains unsolved in general. Nevertheless, the special case of systems driven by dichotomous noise, first considered in the context of activation rates,[14, 15] has received much attention, and exact explicit results were obtained in ref. 16 and later by other authors, using different techniques, in a series of papers.[17, 18] Of all the techniques developed to treat the problem of escape times for systems driven by dichotomous noise (see also ref. 19 for another method valid for more general non-Markovian processes), we think that the stochastic trajectory analysis technique (STAT) presented in ref. 17, which looks directly at the trajectories, is the most transparent, and, as will become clear later, the most useful in considering different boundary conditions.[20,21]

However, trying to apply the above-cited results by Masoliver *et al.*[17] on MFPT to bifurcation problems, we found a certain lack of generality: their derivation cannot be applied to the escape from regions surrounding fixed points (steady states) of the stochastic flow, i.e., states which are stationary for every possible realization of the noise. Moreover, if one calculates the MFPT for regions close to the fixed point x_s and performs the limit of one of the boundaries of the region going to x_s, the MFPT diverges even in situations for which one knows (from an analysis of the stationary probability) that at least a fraction of the trajectories escape from the region.

The aim of this paper is to fill this lack of generality in the calculation of the MFPT for systems driven by dichotomous noise. In order to do this, we find it necessary to split the contributions of the trajectories leaving the region through each of its boundaries. The derivation is presented in Section 2. Section 3 is devoted to the analysis of the behavior of a linear system around a steady state of the stochastic flow. The linear case illustrates the influence of noise on the stability of the steady state and how this stability is reflected in the escape statistics. Finally, we summarize our main conclusions in Section 4.

2. ESCAPE PROBABILITIES AND FIRST PASSAGE TIMES

We consider the one-dimensional general dynamical system

$$\dot{x}_t = F(x_t, \xi_t) \tag{1}$$

where F is a nonlinear function and ξ_t is a symmetric dichotomous noise which can take values $\pm\Delta$ with correlation time $\tau_c = 1/2\lambda$. Time between switches in the noise value, Δ or $-\Delta$, is governed by the distribution $\phi(t) = \lambda \exp(-\lambda t)$, and the average residence time in each of these states is $1/\lambda$.[7] The trajectories of the process x_t can be decomposed into a countable set of time intervals $[t_{n-1}, t_n]$ where the noise remains constant, i.e., only one of the two possible forces $F_+(x) = F(x, \Delta)$ or $F_-(x) = F(x, -\Delta)$ acts upon the system (see Fig. 1). Stochasticity shows up through the time succession $\{t_n\}$ in which the noise changes its value, whereas in each of the time intervals the dynamics is deterministic and autonomous.

The process x lives in the whole real line and we are interested in the (random) time $\tau_{[a,b]}(x_0)$ to go from an initial point $x_0 \in [a, b]$ to one of the boundaries of the interval $[a, b]$. A first simple qualitative analysis shows immediately that if the two forces $F_\pm(x) = F(x, \pm\Delta)$ point in the same direction, say toward b, then the process will leave the region through

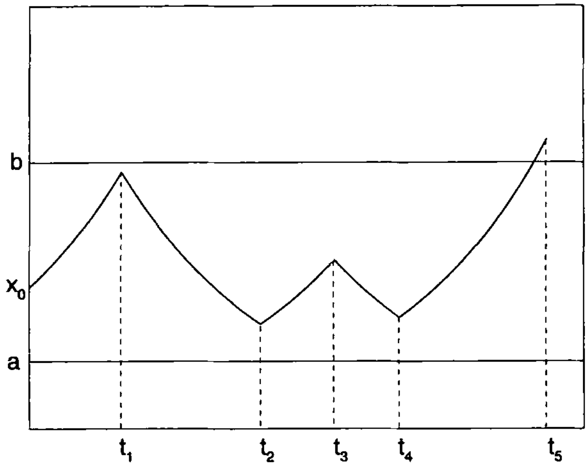

Fig. 1. Typical trajectory of a system perturbed by dichotomous noise.

b in a time between the deterministic times corresponding to both flows F_+ and F_-. The interesting case arises when the forces point in different directions, i.e., regions where the process x_t goes back and forth and therefore the external noise can significantly delay and/or modify the exit from the region[5] (see Fig. 1). Although it is possible to perform the calculation for both cases, we will limit ourselves to regions where $F_+(x)\,F_-(x) \leqslant 0$. To fix ideas, we also choose $F_+(x) \geqslant 0$ and $F_-(x) \leqslant 0$ for the rest of the paper.

The key quantities in our calculation are $f_a(t \mid x_0)\,dt$ and $f_b(t \mid x_0)\,dt$ defined as the probabilities of first reaching, in the time interval $(t, t+dt)$, a or b starting at x_0. We will use superscripts $+$ and $-$ to indicate, when necessary, the initial condition of the noise $\xi(0) = \Delta$ or $\xi(0) = -\Delta$, respectively. Following similar calculations to those performed in ref. 17, one can derive closed equations for the Laplace transform of these distributions

$$\tilde{f}_{a,b}(s \mid x_0) = \int_0^\infty e^{-st} f_{a,b}(t \mid x_0)\,dt. \tag{2}$$

A simple version of the derivation is presented in the appendix. There we prove that, for instance, $\tilde{f}_b^{\pm}(s \mid x_0)$ satisfy

$$\tilde{f}_b^+(s \mid x_0) = e^{-(\lambda+s)\,T_+(x_0 \to b)} + \lambda \int_{x_0}^{b} \frac{dx_1}{F_+(x_1)}\, e^{-(\lambda+s)\,T_+(x_0 \to x_1)} \tilde{f}_b^-(s \mid x_1) \tag{3}$$

[5] There is also another reason for the importance of regions where $F_+(x)\,F_-(x) < 0$: they form the support of the stationary distribution of x. Therefore, these are the relevant regions when calculating escape rates in the stationary regime.

$$\tilde{f}_b^-(s \mid x_0) = -\lambda \int_a^{x_0} \frac{dx_1}{F_-(x_1)} e^{-(\lambda+s)\, T_-(x_0 \to x_1)} \tilde{f}_b^+(s \mid x_1) \tag{4}$$

where

$$T_\pm(x_0 \to x_1) = \int_{x_0}^{x_1} \frac{dx}{F_\pm(x)} \tag{5}$$

is the (deterministic) time to go from x_0 to x_1 under the force $F_\pm(x)$, respectively. These times appear in expressions (3) and (4) as postive quantities, i.e., for every $T_+(x \to y)$, y is bigger than x ($F_+ > 0$) and, conversely, in $T_-(x \to y)$, x is always bigger than y. Notice also that the first term in (3) corresponds, in the time domain, to the delta function

$$e^{-\lambda t}\, \delta(t - T_+(x_0 \to b))$$

which takes into account the escape event without any noise switch. Expressions for the other two distributions are obtained from the former switching simultaneously b to a and $+$ to $-$.

No trouble arises in the former expressions when one considers the escape problem from regions $[a, b]$ where $F_+(x)\, F_-(x)$ is strictly less than zero. Let us discuss now, in the light of Eq. (3) and (4), how the functions $\tilde{f}_{a,b}^\pm(s \mid x_0)$ behave when a or b is a zero of one of the forces. It is easy to check that

$$\tilde{f}_b^+(s \mid x_0) = \tilde{f}_b^-(s \mid x_0) = 0 \qquad \text{if} \quad F_+(b) = 0 \tag{6}$$

and

$$\tilde{f}_a^+(s \mid x_0) = \tilde{f}_a^-(s \mid x_0) = 0 \qquad \text{if} \quad F_-(a) = 0 \tag{7}$$

which is an obvious result since, as we assume that within the interval $[a, b]$, $F_+(x) > 0$ and $F_-(x) < 0$, no trajectory can cross any upper boundary b such that $F_+(b) = 0$, nor a lower boundary a such that $F_-(a) = 0$.

Let us now stress the differences between our approach and that of Masoliver *et al.*[17] They derived a similar integral equation for the Laplace transform of the probability density

$$f^\pm(t \mid x_0) = f_a^\pm(t \mid x_0) + f_b^\pm(t \mid x_0)$$

This is a true probability density, the one corresponding to the escape time through *any* of the boundaries, and it is therefore normalized. On the other hand, $f_{a,b}^\pm(t \mid x_0)$ are not normalized. In fact

$$P_{a,b}^\pm(x_0) = \int_0^\infty dt\, f_{a,b}^\pm(t \mid x_0) = \tilde{f}_{a,b}^\pm(s = 0 \mid x_0) \tag{8}$$

is the probability of leaving the interval through boundary a or b, respectively. The *conditional mean-first-passage times*[6]

$$T^{\pm}_{a,b}(x_0) = \frac{1}{P^{\pm}_{a,b}(x_0)} \int_0^{\infty} dt\, t f^{\pm}_{a,b}(t \mid x_0) = \frac{-1}{P^{\pm}_{a,b}(x_0)} \frac{\partial}{\partial s}\bigg|_{s=0} \tilde{f}^{\pm}_{a,b}(s \mid x_0) \tag{9}$$

are related to the usual MFPT by the expression

$$T^{\pm}(x_0) = P^{\pm}_a(x_0)\, T^{\pm}_a(x_0) + P^{\pm}_b(x_0)\, T^{\pm}_b(x_0) \tag{10}$$

We see here the source of the divergences described in the introduction. If a tends to a fixed point of the stochastic flow x_s, the conditional mean time $T^{\pm}_a(x_0)$ diverges and, if $P^{\pm}_a(x)$ does not tend to zero properly, we end up with an infinite MFPT.

There is an important technical feature of these expressions that we want to remark. If we fix one of the boundaries, say b, and consider a as a variable, the function $\tilde{f}^{\pm}_a$ is continuous with respect to a (and consequently also the corresponding exit probabilities $P^{\pm}_b$, and times $T^{\pm}_b$) even when F_+ or F_- vanishes at this boundary. Likewise, $\tilde{f}^{\pm}_a$, $P^{\pm}_a$, and $T^{\pm}_a$ are continuous functions of b. This allows us to calculate the exit probabilities and times in the neighborhood of a fixed point in a simple way. Consider, for instance, an interval $[a, b]$ with a such that $F_{\pm}(a) = 0$, i.e., a is a steady state, and it is the only one within the interval (there are no other zeros of these forces). To calculate the escape statistics through b we can therefore consider an interval of the form $[a + \varepsilon, b]$, calculate P_b and T_b, and then perform the limit $\varepsilon \to 0$.

Equations (3) and (4) are equivalent to second-order linear differential equations with boundary conditions. In fact, both functions $\tilde{f}_a(s \mid x_0)$ and $\tilde{f}_b(s \mid x_0)$ satisfy the same equation but with different boundary conditions. For instance, if the initial condition for the noise is $+\Delta$, this equation reads

$$\left[\frac{\partial^2}{\partial x_0^2} + \left[\frac{F'_+(x_0)}{F_+(x_0)} - \frac{s+\lambda}{F_+(x_0)} - \frac{s+\lambda}{F_-(x_0)}\right]\frac{\partial}{\partial x_0} + \frac{s(s+2\lambda)}{F_+(x_0)\, F_-(x_0)}\right] \tilde{f}^{+}_{a,b}(s \mid x_0) = 0 \tag{11}$$

and the boundary conditions are

$$\tilde{f}^{+}_a(s \mid b) = 0$$

$$\frac{\partial}{\partial x}\bigg|_{x=a} \tilde{f}^{+}_a(s \mid x) = \frac{-\lambda + (s+\lambda)\, \tilde{f}^{+}_a(s \mid a)}{F_+(a)} \tag{12}$$

[6] It is important to realize that *conditional* means that $T^{\pm}_{a,b}$ only count the time taken by the trajectories that actually reach the boundary a or b, respectively.

and

$$\tilde{f}_b^+(s \mid b) = 1$$

$$\left.\frac{\partial}{\partial x}\right|_{x=a} \tilde{f}_b^+(s \mid x) = \frac{(s+\lambda)\, \tilde{f}_b^+(s \mid a)}{F_+(a)} \tag{13}$$

If the initial condition for the noise is $-\Delta$, these expressions are the same switching F_+ to F_- and a to b. Notice that these are singular boundary conditions if a is such that $F_+(a) = 0$. Nevertheless, due to the continuity argument stated before, $\tilde{f}_a(s \mid b)$ and $\tilde{f}_b(s \mid a)$ are well behaved and we can obtain from them the corresponding escape probabilities and conditional first passage times. In other words, we can use the differential equations with boundaries $\tilde{a}$ and $\tilde{b}$, for which the forces $F_\pm$ do not vanish, and then perform the limits $\tilde{a} \to a$ and $\tilde{b} \to b$.

The solution of the differential equation (11) can be easily found for the special case $s = 0$, and this solution is precisely the escape probability (8),

$$P_a^+(x_0) = \frac{\lambda}{1 + \lambda g^+(a)}\, g^+(x_0) \tag{14}$$

$$P_b^+(x_0) = 1 - P_a^+(x_0) \tag{15}$$

with

$$g^+(x_0) = \int_{x_0}^{b} dy\, \frac{1}{F_+(y)} \exp\left[\lambda \int_a^y ds \left(\frac{1}{F_+(s)} + \frac{1}{F_-(s)}\right)\right] \tag{16}$$

As always, the corresponding expressions for $P_{a,b}^-(x_0)$ (initial condition for the noise $-\Delta$) are obtained switching a to b and $+$ to $-$.

The conditional MFPT is calculated differentiating Eq. (11) with respect to s and setting $s = 0$. The solution reads

$$T_{a,b}^{\pm}(x_0) = \frac{1}{P_{a,b}^{\pm}(x_0)} \left[\int_b^{x_0} dy\, V_{a,b}^{\pm}(y)\, e^{-M^\pm(y)} + C \int_b^{x_0} dy\, e^{-M^\pm(y)}\right] \tag{17}$$

where

$$M^\pm(x) = \int^x dy \left[\frac{F'_\pm(y)}{F_\pm(y)} - \lambda\left(\frac{1}{F_+(y)} + \frac{1}{F_-(y)}\right)\right] \tag{18}$$

$$V_{a,b}^{\pm}(x) = \int^x dy\, e^{M^\pm(y)} \left[\frac{2\lambda}{F_+(y)\, F_-(y)} - \left(\frac{1}{F_+(y)} + \frac{1}{F_-(y)}\right)\frac{\partial}{\partial y}\right] P_{a,b}^{\pm}(y) \tag{19}$$

and the constant C is fixed by the boundary conditions

$$\left.\frac{d}{dx}\right|_{x=b} [P^-_{a,b} T^-_{a,b}](x) = \frac{\lambda[P^-_{a,b} T^-_{a,b}](b) - P^-_{a,b}(b)}{F_-(b)} \tag{20}$$

$$\left.\frac{d}{dx}\right|_{x=a} [P^+_{a,b} T^+_{a,b}](x) = \frac{\lambda[P^+_{a,b} T^+_{a,b}](a) - P^+_{a,b}(a)}{F_+(a)} \tag{21}$$

Equations (14) and 17 are the main results of this work, and together with the calculation of the stationary probability distributions, following the method explained in detail in ref. 7, solve completely the problem of calculating the behavior of one-dimensional nonlinear systems perturbed by dichotomous noise.

A comment on the boundary conditions (b.c.) is in order. It should be clearly understood that the conditions (12) and (13) come directly from (3) and (4), i.e., are "natural" b.c. given by the dynamics of the system under consideration. They therefore include not only the absorbing b.c. considered in most work on this topic,[15-18] but also the "fixed-point b.c.", when one of the boundaries of the interval is a steady state of the stochastic flow and cannot be reached, and any other situation in which there is not a particular behavior at the boundaries imposed from outside the system.

The situation is different, for instance, when we are dealing with reflecting (instantaneous or delayed) boundaries.[20-21] However, it is remarkable that our general results for $f^{\pm}_{a,b}(t \mid x_0)$ with natural b.c. can be used to study those relevant situations of reflecting or even mixed boundary conditions. Let us consider, for instance, that trajectories reaching the lower boundary a may be transmitted or reflected according to waiting time probability distributions $\phi_T(t)$ and $\phi_R(t)$ respectively (the sum of the time integral of these distributions is obviously normalized), whereas boundary b remains natural (absorbing), and let $\psi^{\pm}_{a,b}(t \mid x_0)$ be the new escape time probability distributions (the equivalents of our previous $f^{\pm}_{a,b}$), which are the fundamental quantities for obtaining the escape probabilities and times. Now the event "escape through boundary b" can occur in two ways: the system escapes without touching a, or the system reaches a, is reinjected, and then leaves through b. These two possibilities bring about terms involving convolutions of the time distributions. With a similar reasoning for the escape through a, and after Laplace transforming, the expressions that give $\tilde{\psi}^{\pm}_{a,b}(s \mid x_0)$ become algebraic:

$$\tilde{\psi}^{\pm}_b(s \mid x_0) = \tilde{f}^{\pm}_b(s \mid x_0) + \tilde{f}^{\pm}_a(s \mid x_0)\, \tilde{\phi}_R(s)\, \psi^{+}_b(s \mid a) \tag{22}$$

$$\tilde{\psi}^{\pm}_a(s \mid x_0) = \tilde{f}^{\pm}_a(s \mid x_0)\, \tilde{\phi}_T(s) + \tilde{f}^{\pm}_a(s \mid x_0)\, \tilde{\phi}_R(s)\, \tilde{\psi}^{+}_a(s \mid a) \tag{23}$$

where the quantities $\tilde{f}^{\pm}_{a,b}$ are the solutions of (11) with natural b.c. (12) and (13). Setting $x_0 = a$ in (22) and (23), we easily calculate $\tilde{\psi}^+_b(s \mid a)$ and $\tilde{\psi}^+_a(s \mid a)$, yielding the final expressions

$$\tilde{\psi}^{\pm}_b(s \mid x_0) = \tilde{f}^{\pm}_b(s \mid x_0) + \tilde{f}^{\pm}_a(s \mid x_0) \frac{\tilde{\phi}_R(s)\, \tilde{f}^+_b(s \mid a)}{1 - \tilde{\phi}_R(s)\, \tilde{f}^+_a(s \mid a)} \tag{24}$$

and

$$\tilde{\psi}^{\pm}_a(s \mid x_0) = \frac{\tilde{\phi}_T(s)\, \tilde{f}^{\pm}_a(s \mid x_0)}{1 - \tilde{\phi}_R(s)\, \tilde{f}^+_a(s \mid a)} \tag{25}$$

For instantaneous and perfect reflection, i.e., $\phi_T(t) = 0$, $\phi_R(t) = \delta(t)$, and $\tilde{\phi}_R(s) = 1$, we recover the results in ref. 20, whereas if the trajectories that reach the boundary a are reinjected in the interval when the noise value changes, that is, $\phi_T(t) = 0$, $\phi_R(t) = \lambda \exp(-\lambda t)$, and $\tilde{\phi}_R(s) = \lambda(\lambda + s)^{-1}$, we get those in ref. 21. More complicated cases when both transmission and reflection are possible could be analyzed from (24) and (25).

3. LINEAR DYNAMICS

Most of the aspects discussed so far can be best illustrated by analyzing the linear case. We will study in this section the escape statistics of a linear system from the neighborhood of a steady state.

Consider the system

$$\dot{x}_t = (c + \xi_t)\, x_t \tag{26}$$

for which $x_s = 0$ is a steady state, i.e., a zero of the two forces $F_{\pm} = c \pm \Delta$. We concentrate our attention on the escape from an interval $[0, b]$. It is convenient to use the notation $\alpha = 1/(c + \Delta)$ and $\beta = -1/(c - \Delta)$, so that $F_+(x) = x/\alpha$ and $F_-(x) = -x/\beta$ with $\alpha, \beta > 0$. With this, the quantities $\tilde{f}^{\pm}_b(s \mid x_0)$ satisfy the same equation independently of the initial condition of the noise

$$\left[x_0^2 \frac{\partial^2}{\partial x_0^2} + x_0[1 - (s + \lambda)(\alpha - \beta)] \frac{\partial}{\partial x_0} - s(s + 2\lambda)\, \alpha\beta \right] \tilde{f}_b(s \mid x_0) = 0 \tag{27}$$

The general solution of this equation is

$$\tilde{f}_b(s \mid x_0) = A(s)\, x_0^{r_1} + B(s)\, x_0^{r_2} \tag{28}$$

where

$$r_{1,2}=\frac{(s+\lambda)(\alpha-\beta)}{2}\pm\frac{1}{2}[(s+\lambda)^2(\alpha+\beta)^2-4\alpha\beta\lambda^2]^{1/2} \tag{29}$$

Therefore, for this linear case, we can explicitly obtain the general expression for the Laplace transform of the probability density. The functions $A(s)$ and $B(s)$ are determined by the boundary conditions (12) and (13).

Since (28) with $A(s)$ and $B(s)$ calculated as indicated gives indeed the solution for the interval $[a, b]$, we now use the continuity property and take the limit $a \to 0$ to obtain

$$\tilde{f}_b^+(s \mid x_0)=\left(\frac{x_0}{b}\right)^{r_1(s)} \tag{30}$$

$$\tilde{f}_b^-(s \mid x_0)=\frac{\beta\lambda}{r_1(s)+\beta(s+\lambda)}\left(\frac{x_0}{b}\right)^{r_1(s)} \tag{31}$$

which are the expressions that contain all the statistics of the problem. Notice that for the critical case $\alpha=\beta$, the exponent $r_1=\alpha[s(s+2\lambda)]^{1/2}$ and $\tilde{f}_b(s \mid x_0)$ is not analytic at $s=0$, leading to an infinite MFPT.

Following the procedure described in the previous section, we get

$$P_b^+(x_0)=\begin{cases}(x_0/b)^{\lambda(\alpha-\beta)}, & \alpha>\beta\\ 1, & \alpha\leqslant\beta\end{cases} \tag{32}$$

$$P_b^-(x_0)=\begin{cases}(\beta/\alpha)(x_0/b)^{\lambda(\alpha-\beta)}, & \alpha>\beta\\ 1, & \alpha\leqslant\beta\end{cases} \tag{33}$$

This latter probability can be factorized into a contribution from the first return to the initial condition $P_{x_0}^-(x_0)$—which does not depend on λ—and a further motion of x from x_0 to b:

$$P_b^-(x_0)=P_{x_0}^-(x_0)\,P_b^+(x_0)=\frac{\beta}{\alpha}P_b^+(x_0)$$

Averaging now over the two possible initial conditions of the noise, we finally get

$$P_b(x_0)=\begin{cases}[(\alpha+\beta)/2\alpha](x_0/b)^{\lambda(\alpha-\beta)}, & \alpha>\beta\\ 1, & \alpha\leqslant\beta\end{cases} \tag{34}$$

Figure 2a shows the probabilities of escape for different values of λ and $\alpha-\beta=1$. Three characteristic times can be distinguished: α, β, and the

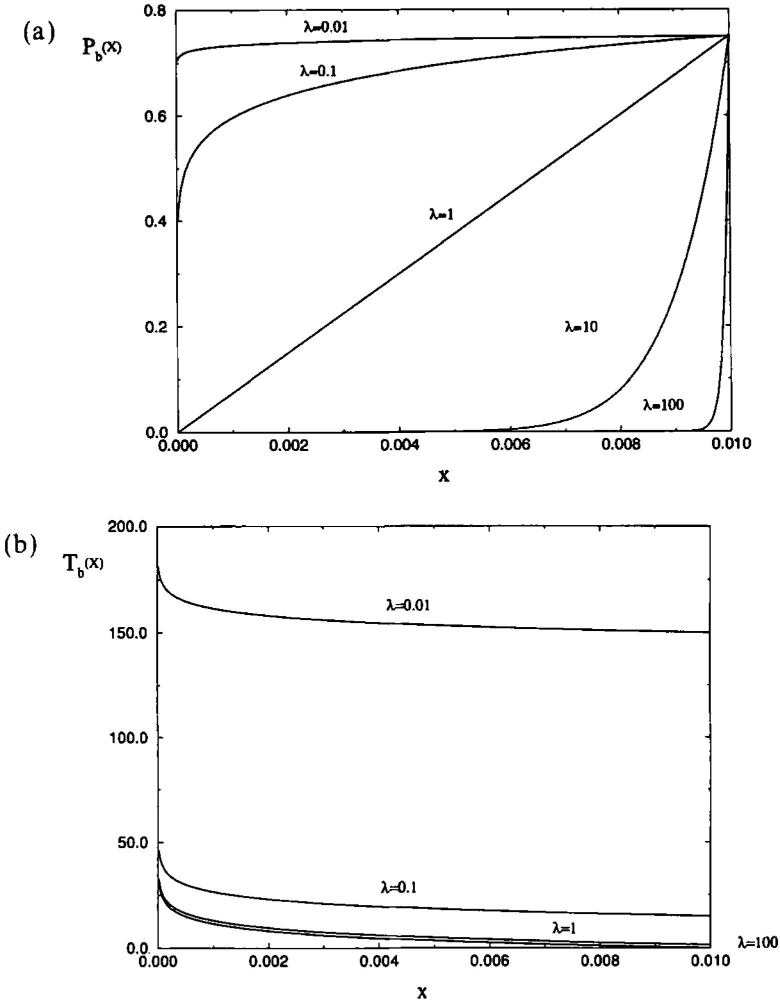

Fig. 2. (a) Escape probabilities and (b) mean escape times for the linear system with $\alpha = 2$, $\beta = 1$, and different values of the correlation time of the noise.

one associated with the noise, $1/\lambda$. For a very slow noise, i.e., $1/\lambda \gg \alpha - \beta$, trajectories with initial condition $+\Delta$ escape immediately without any switch of the noise value. If the initial condition is $-\Delta$, the probability of escape will tend to the probability of return to the initial value. It is remarkable that this return probability is different from zero, because the lower limit of the interval is the steady state 0, and, for large but finite $1/\lambda$, the noise will switch with probability one. The limits $\lambda \to 0$ and $a \to 0$

obviously do not commute: if one first sets $\lambda \to 0$ keeping a small but different from zero, then the noise becomes constant in time and therefore $P_b^-(x_0)=0$, whereas from (33) we see that $a \to 0$ followed by $\lambda \to 0$ gives $P_b^-(x_0)=\beta/\alpha$. On the other hand, if $\lambda \to \infty$, one recovers the deterministic system. This is so since the dichotomous noise becomes a white noise when simultaneously λ and Δ go to infinity,[7] but if the correlation time is set equal to zero while keeping Δ finite, one ends up with a white noise of zero intensity, obtaining $P_b(x_0)=0$ for the values of α and β corresponding to Fig. 2a.

Now, by differentiating $\tilde{f}_b^{\pm}$ we obtain the conditional MFPT of crossing b:

$$T_b^+(x_0)=\begin{cases}\dfrac{\alpha^2+\beta^2}{\alpha-\beta}\log\dfrac{b}{x_0}, & \alpha>\beta\\[2mm] \dfrac{2\alpha\beta}{\beta-\alpha}\log\dfrac{b}{x_0}, & \alpha<\beta\end{cases} \tag{35}$$

$$T_b^-(x_0)=\begin{cases}\dfrac{\alpha^2+\beta^2}{\alpha-\beta}\log\dfrac{b}{x_0}+\dfrac{\alpha+\beta}{\lambda(\alpha-\beta)}, & \alpha>\beta\\[2mm] \dfrac{2\alpha\beta}{\beta-\alpha}\log\dfrac{b}{x_0}+\dfrac{\alpha+\beta}{\lambda(\beta-\alpha)}, & \alpha<\beta\end{cases} \tag{36}$$

Notice that the escape times diverge to infinity at the bifurcation point $\alpha=\beta$. This divergence is interpreted as a true critical slowing down: all trajectories escape with probability one, but, on average, they take an infinite time. It is easy to see that the constant terms in T_b^- are the mean times needed to come back and cross the initial point x_0. Since this can only occur when the noise value is $+\Delta$, the nonconstant terms are the same in T_b^+ and T_b^-, and these are precisely the terms that do not depend on λ.

Averaging over the two possible initial conditions of the noise, we obtain (Fig. 2b)

$$T_b(x_0)=\begin{cases}\dfrac{\alpha^2+\beta^2}{\alpha-\beta}\log\dfrac{b}{x_0}+\dfrac{\alpha+\beta}{2\lambda(\alpha-\beta)}, & \alpha>\beta\\[2mm] \dfrac{2\alpha\beta}{\beta-\alpha}\log\dfrac{b}{x_0}+\dfrac{\alpha+\beta}{2\lambda(\beta-\alpha)}, & \alpha<\beta\end{cases} \tag{37}$$

The case $\alpha>\beta$ is remarkable since the behavior of the mean escape time is somehow counterintuitive. Consider, for instance, the term in $T_b^-(x_0)$ independent of x_0, i.e., $T_{x_0}^-(x_0)=(\alpha+\beta)/\lambda(\alpha-\beta)$. In terms of the original variables of the system, $\alpha>\beta$ means $c<0$, and $T_{x_0}^-(x_0)=\Delta/\lambda\,|c|$. Increasing $|c|$, the tendency to move toward zero also increases and therefore one would expect that the time to return to x_0 would become larger.

On the contrary, $T^-_{x_0}(x_0)$ decreases!. Recalling that $T^-_{x_0}(x_0)$ is by definition a *conditional* average, this result indicates that, for large $|c|$, the trajectories that actually return to x_0 are those which do not move very far from x_0.

4. CONCLUSIONS

We have extended the existing results on escape statistics under the action of a dichotomous noise to situations including the existence of true stationary points of the forces acting upon the system, i.e., fixed points of the stochastic flow, and different boundary conditions. With this we think that the one-dimensional case is completely solved.

Using our general results on exit probabilities and the stability of the fixed points, we can draw the bifurcation diagram for *all one-dimensional ssystems*. The advantage of using the dichotomous noise is that we obtain the exit probabilities and the conditional mean first passage times analytically, and this, together with the calculation of the stationary probability distributions with support in the invariant sets, leads to a complete description, for the first time for a nonwhite noise, of the evolution of the stochastic system.

In that sense we claim that, for many practical purposes, the asymptotic study (mainly the bifurcation diagrams and the stationary densities) is not enough to describe stochastic systems, and consider that a knowledge of escape probabilities and exit times is essential for a full understanding of th behavior of the system.

APPENDIX

Equations (3) and (4) can be derived using a recurrence argument. Let $f_b^+(t \mid x_0; n)$ be the probability of escaping through b after n noise switches with initial condition for the noise $+\Delta$. From Fig. 1 it is not difficult to write the following recurrence equation, for all $n \geqslant 1$:

$$f_b^+(t \mid x_0; 2n) = \int_0^{T_+(x_0 \to b)} dt_1\, \lambda e^{-\lambda t_1} f_b^-(t - t_1 \mid \phi_+^{t_1}(x_0); 2n-1)\, \theta(t - t_1) \tag{A1}$$

where $\theta(x)$ is the Heaviside step function, $\phi'_+(x)$ is the flow or time evolution induced by F_+, and

$$T_+(x \to y) = \int_x^y \frac{dx'}{F_+(x')}$$

is the time to go from x to y under this flow.

Taking the Laplace transform of (A1)

$$\tilde{f}_b^+(s \mid x_0; 2n) = \int_0^{T_+(x_0 \to b)} dt_1\, \lambda e^{-(\lambda+s)t_1} \tilde{f}_b^-(s \mid \phi_+^{t_1}(x_0); 2n-1) \quad \text{(A2)}$$

and with the change of variable $x_1 = \phi_+^{t_1}(x_0)$ one finally has

$$\tilde{f}_b^+(s \mid x_0; 2n) = \int_{x_0}^{b} \frac{dx_1}{F_+(x_1)} \lambda e^{-(\lambda+s)\, T_+(x_0 \to x_1)} \tilde{f}_b^-(s \mid x_1; 2n-1) \quad \text{(A3)}$$

The probability of escaping before any noise switch is easily calculated as

$$f_b^+(t \mid x_0; 0) = e^{-\lambda t}\, \delta(t - T_+(x_0 \to b)) \quad \text{(A4)}$$

and summing (A3) over n plus the Laplace transform of (A4), one obtains (3). A similar argument for $f_b^-(t \mid x_0)$ leads to Eq. (4).

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REFERENCES

1. J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcation of Vector Fields* (Springer-Verlag, Berlin, 1983).
2. R. Z. Hasmiskii, *Stochastic Stability of Differential Equations* (Sijthoff and Noordhoff, Alphen aan den Rijn, 1980)
3. M. Lücke, In *Noise in Nonlinear Dynamical Systems*, Vol. 2, F. Moss and P. V. E. McClintock, eds. (Cambridge University Press, Cambridge, 1989).
4. R. Graham, and A. Schenzle, *Phys. Rev. A* **26**:1676 (1982).
5. V. I. Klyatskin, *Radiophys. Quantum Electron.* **20**:382 (1977).
6. E. Knobloch and K. Wiesenfeld, *J. Stat. Phys.* **33**:611 (1983).
7. W. Horsthemke and R. Lefever, *Noise-Induced Transitions* (Springer-Verlag, Berlin, 1984).
8. M. C. Mackey, A. Longtin, and A. Lasota, *J. Stat. Phys.* **60**:735 (1990).
9. K. H. Hoffmann, *Z. Phys. B* **49**:245 (1982).
10. C. Meunier and A. D. Verga, *J. Stat. Phys.* **50**:345 (1988).
11. J. C. Nuño, F. Montero, and F. J. de la Rubia, *J. Theor. Biol.* **165**:553 (1993).
12. W. Kliemann, *Bull. Math. Biol.* **45**:483 (1983)
13. J. M. Sancho, M. San Miguel, L. Pesquera, and M. A. Rodriguez, *Physica A* **142**:532 (1987).
14. P. Hänggi and P. Riseborough, *Phys. Rev. A* **27**:3379 (1983).
15. C. Van den Broeck and P. Hänggi, *Phys. Rev. A* **30**:2730 (1984).
16. P. Hänggi and P. Talkner, *Phys. Rev. A* **32**:1934 (1985).
17. J. Masoliver, K. Lindenberg, and B. J. West, *Phys. Rev. A* **33**:2177 (1986); **34**:1481 (1986); **34**:2351 (1986).
18. M. A. Rodríguez and L. Pesquera, *Phys. Rev. A* **34**:4532 (1986).
19. G. H. Weiss, J. Masoliver, K. Lindenberg, and B. J. West, *Phys. Rev. A* **36**:1435 (1987).
20. V. Balakrishnan, C. Van den Broeck, and P. Hänggi, *Phys. Rev. A* **38**:4213 (1988).
21. U. Behn and K. Schiele, *Z. Phys. B* **77**:485 (1989).